MODULI OF CURVES AND SPIN STRUCTURES VIA ALGEBRAIC GEOMETRY

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Abstract. Here we investigate some birational properties of two collections of moduli spaces, namely moduli spaces of (pointed) stable curves and of (pointed) spin curves. In particular, we focus on vanishings of Hodge numbers of type $(p,0)$ and on computations of the Kodaira dimension. Our methods are purely algebro-geometric and rely on an induction argument on the number of marked points and the genus of the curves (cf. [3]).

1. Introduction

In the last decade, ideas from physics have reinvigorated the interest in moduli spaces. As a consequence, the problem of investigating their geometry has become more and more intriguing. Usually, such spaces parametrize (pointed) genus g complex curves with, possibly, extra structures defined on C, such as a morphism to a fixed variety or a spin structure. Moreover, they may be usually viewed as objects in different categories, specifically as stacks, orbifolds, or $\mathbb{Q}$-factorial projective schemes.

In the present paper, we focus on two classical moduli spaces, namely $\overline{\mathcal{M}}_{g,n}$ and $\overline{S}_{g,n}$. We recall that the former one parametrizes n-pointed genus g stable curves and the latter one parametrizes n-pointed quasi-stable curves C of genus g with a line bundle whose second tensor power is isomorphic to the dualizing sheaf ω_C: see Section 3 for precise definitions. Throughout, we regard them as projective normal varieties over the field of complex numbers.

It is impossible to condense in a short paragraph all the various results and open problems in connection with these two varieties. We refer the reader to, for instance, [10], [11], [21] and the references cited therein. We just mention here that some birational invariants of $\overline{\mathcal{M}}_{g,n}$ are only conjecturally known, like the Kodaira dimension of $\overline{\mathcal{M}}_g$, $17 \leq g \leq 23$, or partially computed, like $(p,0)$-type Hodge numbers (cfr. [13], [14], [15], and [27]). Furthermore, little is known about $\overline{S}_{g,n}$. As a finite ramified covering of $\overline{\mathcal{M}}_{g,n}$, its geometry is even more complicated. Recent results on the topology and the rational cohomology can be found, for example, in [7], [8], [9], [19], [20]. For a different, stack-theoretic approach see [1], [23], [24].

As shown by Enrico Arbarello and Maurizio Cornalba in [3], a purely algebro-geometric investigation of $\overline{\mathcal{M}}_{g,n}$ becomes more natural when the whole collection of $\overline{\mathcal{M}}_{g,n}$'s, and specific morphisms among them, is taken into account. By an elementary double induction argument, it is thus possible to prove various results on the rational cohomology of these spaces: see [17], [18] for related results which had been previously known via other methods.

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Here we apply a similar induction argument to $\overline{\mathcal{M}}_{g,n}$ and $\overline{S}_{g,n}$. As a result, in Section 2 we derive some vanishings of Hodge numbers of type $(p,0)$ for $\overline{\mathcal{M}}_{g,n}$. Next, in Section 3 we describe the rational Picard group of $\overline{S}_{g,n}$ and, as a corollary of [20], we determine generators and relations in the case $g \geq 9$, $n = 0$. In Section 4, we compute the Kodaira dimension of spin moduli spaces in several cases and leave the other ones as open questions, which we hope to address in the future with different methods. To carry through some of these calculations, we also compute the Kodaira dimension of $\overline{\mathcal{M}}_{1,n}$.

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2. $(p,0)$ Hodge Numbers of Moduli Spaces of Stable Curves

In [3], Enrico Arbarello and Maurizio Cornalba proved the vanishing of some odd cohomology groups of $\overline{\mathcal{M}}_{g,n}$. They applied an inductive method that reduces the problem to checking such vanishings for finitely many values of g and n in each odd degree k. Unfortunately, if $k \geq 11$ the inductive machinery does not work: indeed, it is well known (see, for instance, [27], proof of Corollary 4.7, or [15], Proposition 2) that $h^{11,0}(\overline{\mathcal{M}}_{1,11}) \neq 0$, where $h^{p,q}(\overline{\mathcal{M}}_{g,n}) = \dim H^{p,q}(\overline{\mathcal{M}}_{g,n})$ denote the Hodge numbers of $\overline{\mathcal{M}}_{g,n}$. On the other hand, from the results of [3] it follows that $h^{p,0}(\overline{\mathcal{M}}_{g,n}) = 0$ for $p = 1, 3, 5$. Here we complete the picture by showing that there are no nonzero p holomorphic forms on $\overline{\mathcal{M}}_{g,n}$ for $0 < p < 11$. Precisely, the following holds.

Theorem 1. *Let g and n be non-negative integers, $n > 2-2g$. If $0 < p < 11$, then $h^{p,0}(\overline{\mathcal{M}}_{g,n}) = 0$.*

Proof. As in [3], we use the long exact sequence of cohomology with compact supports:

$$\ldots \to H^k_c(\mathcal{M}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n}) \to \ldots$$

and the vanishing of $H^k_c(\mathcal{M}_{g,n})$ for

$$k \leq d(g,n) = \begin{cases} n-4 & \text{if } g = 0 \\ 2g-2 & \text{if } n = 0 \\ 2g-3+n & \text{if } g > 0, n > 0. \end{cases}$$

Moreover, since the morphism

$$H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n})$$

is compatible with the Hodge structures (see [3], p. 102), there is an injection

$$H^{p,0}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^{p,0}(\partial\mathcal{M}_{g,n})$$

for $p \leq d(g,n)$.

In order to conclude, it suffices to check that $h^{p,0}(\overline{\mathcal{M}}_{g,n}) = 0$ for every $p > d(g,n)$. This is in fact the base of the inductive procedure. When $g = 0$, the space $\overline{\mathcal{M}}_{0,n}$ is rational [25]; so there is nothing to check. If $g = 1$, we are dealing with $\overline{\mathcal{M}}_{1,n}$ for $n \leq 10$. As shown in [5], all these spaces are rational, so the base of the induction holds true. Finally, when $g \geq 2$ and

$p > d(g,n)$, $\overline{\mathcal{M}}_{g,n}$ is a unirational variety - see [26], Theorem 7.1). Hence, from a dominant rational map

$$\mathbb{P}^{3g-3+n} \dashrightarrow \overline{\mathcal{M}}_{g,n}$$

we get an injective morphism

$$H^{p,0}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^{p,0}(\mathbb{P}^{3g-3+n})$$

as in [16], p. 494, and we have $h^{p,0}(\mathbb{P}^{3g-3+n}) = 0$ for every $p > 0$ (see for instance [16], Corollary on p. 118). Thus the claim is completely proved.

$\square$

3. The Rational Picard Group of Spin Moduli Spaces

The moduli space S_g of smooth spin curves is a classical object, which parametrizes pairs given by (smooth genus g complex curve C, and theta-characteristic on C). Since it is a non-trivial covering of $\mathcal{M}_g$, *a priori* its geometry is much more complicated. However, John Harer in [19] and [20] succeeded in applying to S_g his approach to the cohomology of moduli spaces via geometric topology. Furthermore, in [8] and [9], Maurizio Cornalba constructed a geometrically meaningful compactification $\overline{S}_g$ of S_g. Here we determine $\mathrm{Pic}(\overline{S}_{g,n}) \otimes \mathbb{Q}$ and give explicit generators and relations. For reader's convenience we recall some basic definitions.

Let X be a Deligne-Mumford semistable curve and let E be a complete, irreducible subcurve of X. E is said to be *exceptional* when it is smooth, rational, and meets the other components in exactly two points. Moreover, X is said to be *quasi-stable* when any two distinct exceptional components of C are disjoint. In the sequel, $\tilde{X}$ will denote the subcurve $\overline{X \setminus \cup E_i}$ obtained from X by removing all exceptional components.

A *spin curve* of genus g (see [8], § 2) is the datum of a quasi-stable genus g curve X with an invertible sheaf ζ_X of degree $g-1$ on X and a homomorphism of invertible sheaves

$$\alpha_X : \zeta_X^{\otimes 2} \longrightarrow \omega_X$$

such that

(i) ζ_X has degree 1 on every exceptional component of X;

(ii) α_X is not zero at a general point of every non-exceptional component of X.

Therefore, α_X vanishes identically on all exceptional components of X and induces an isomorphism

$$\tilde{\alpha}_X : \zeta_X^{\otimes 2}|_{\tilde{X}} \longrightarrow \omega_{\tilde{X}}.$$

In particular, when X is smooth, ζ_X is just a theta-characteristic on X.

Two spin curves (X, ζ_X, α_X) and $(X', \zeta_{X'}, \alpha_{X'})$ are *isomorphic* if there are isomorphisms $\sigma : X \to X'$ and $\tau : \sigma^*(\zeta'_X) \to \zeta_X$ such that τ is compatible with the natural isomorphism between $\sigma^*(\omega_{X'})$ and ω_X.

A *family of spin curves* is a flat family of quasi-stable curves $f : \mathcal{X} \to S$ with an invertible sheaf ζ_f on $\mathcal{X}$ and a homomorphism

$$\alpha_f : \zeta_f^{\otimes 2} \longrightarrow \omega_f$$

such that the restriction of these data to any fiber of f gives rise to a spin curve.
Two families of spin curves $f : \mathcal{X} \to S$ and $f' : \mathcal{X}' \to S$ are *isomorphic* if there are isomorphisms $\sigma : \mathcal{X} \longrightarrow \mathcal{X}'$ and $\tau : \sigma^*(\zeta_{f'}) \longrightarrow \zeta_f$ such that $f = f' \circ \sigma$ and τ is compatible with the natural isomorphism between $\sigma^*(\omega_{f'})$ and ω_f.
Let $\overline{S}_g$ be the set of isomorphism classes of spin curves of genus g and S_g be the subset consisting of classes of smooth curves. One can define a natural structure of analytic variety on $\overline{S}_g$ (see [8], §5) in such a way that for any spin curve X there is a neighbourhood of $[X]$ in $\overline{S}_g$ and an isomorphism:

$$U \cong B_X/\mathrm{Aut}(X),$$

where B_X is a $3g-3$-dimensional polydisk and $\mathrm{Aut}(X)$, the group of automorphisms of the spin curve X, is a finite group (see [8], Lemma (2.2)). These spaces can be slightly generalized as follows:

$$\begin{aligned}\overline{S}_{g,n} \quad := \quad & \{[(C, p_1, \dots, p_n; \zeta; \alpha)] : (C, p_1, \dots, p_n) \text{ is a genus } g \\ & \text{quasi-stable projective curve with } n \text{ marked points;} \\ & \zeta \text{ is a line bundle of degree } g-1 \text{ on } C \text{ having} \\ & \text{degree 1 on every exceptional component of } C\text{, and} \\ & \alpha : \zeta^{\otimes 2} \to \omega_C \text{ is a homomorphism which is} \\ & \text{not zero at a general point of every non-exceptional} \\ & \text{component of } C\}.\end{aligned}$$

Analogously to $\overline{S}_g$, these spaces are normal projective varieties of complex dimension $3g-3+n$ with finite quotient singularities. We point out the following fact:

Lemma 1. *Let* $\mathrm{Pic}(\overline{S}_{g,n}) := H^1(\overline{S}_{g,n}, \mathcal{O}^*)$*. There is a natural isomorphism*

$$\mathrm{Pic}(\overline{S}_{g,n}) \otimes \mathbb{Q} \xrightarrow{\cong} A_{3g-4+n}(\overline{S}_{g,n}) \otimes \mathbb{Q}.$$

Proof. Since $\overline{S}_{g,n}$ is normal (see [8], Proposition (5.2)), there is an injection:

$$\mathrm{Pic}(\overline{S}_{g,n}) \hookrightarrow A_{3g-4+n}(\overline{S}_{g,n}).$$

Moreover, from the construction of $\overline{S}_{g,n}$ it follows that the singularities of $\overline{S}_{g,n}$ are of finite quotient type, so every Weil divisor is $\mathbb{Q}$-Cartier and there is a surjective morphism:

$$\mathrm{Pic}(\overline{S}_{g,n}) \otimes \mathbb{Q} \twoheadrightarrow A_{3g-4+n}(\overline{S}_{g,n}) \otimes \mathbb{Q}.$$

Hence the claim follows. □

When $n = 0$, it is also possible to give a more precise description of of $\mathrm{Pic}(\overline{S}_g) \otimes \mathbb{Q}$. To this end, recall that $\overline{S}_g$ is the disjoint union of two irreducible subvarieties $\overline{S}_g^+$ and $\overline{S}_g^-$ which consist of the even and the odd spin curves of genus g (see [8], Lemma (6.3)), respectively. The following crucial result was obtained by John Harer via geometric topology (see [20], Corollary 1.3):

Theorem 2. *Let $\mathcal{M}_g[\varepsilon]$ denote either $\overline{S}_g^+ \cap S_g$ or $\overline{S}_g^- \cap S_g$. Then* $\mathrm{Pic}(\mathcal{M}_g[\varepsilon]) \otimes \mathbb{Q} := H^1(\mathcal{M}_g[\varepsilon], \mathcal{O}^*) \otimes \mathbb{Q}$ *has rank 1 for $g \geq 9$.*

For any family $f : \mathcal{X} \to S$ of spin curves, $M_f := \det Rf_* \zeta_f$ is a line bundle on S. Let M denote the corresponding line bundle on $\overline{S}_g$ associated to the universal family on $\overline{S}_g$ (as usual, notice that for $g \geq 4$ the locus of spin curves with automorphisms has complex codimension ≥ 2). Let μ^+ (resp. μ^-) be the class of M in $\mathrm{Pic}(\overline{S}_g^+)$ (resp. $\mathrm{Pic}(\overline{S}_g^-)$. The boundary $\partial \overline{S}_g = \overline{S}_g \setminus S_g$ is the union of irreducible components A_i^+, B_i^+ (contained in $\overline{S}_g^+$) and A_i^-, B_i^- (contained in $\overline{S}_g^-$), which are completely described in [8], § 7. Let α_i^+, β_i^+, α_i^-, β_i^- denote the corresponding classes in $A_{3g-4}(\overline{S}_g)$. The following result is contained in [8], Proposition (7.2):

Proposition 1. *If $g > 2$ is odd, the classes μ^+, μ^-, α_i^+, α_i^-, β_i^+, β_i^-, $i = 0, \dots, (g-1)/2$, are independent. If $g > 2$ is even, the same is true of the classes μ^+, μ^-, α_i^+, α_i^-, β_i^+, $i = 0, \dots, g/2$, and β_i^-, $i = 0, \dots, g/2 - 1$.*

Now we are ready to state and prove a description by generators and relations of the rational Picard group of $\overline{S}_g$:

Corollary 1. *Assume $g \geq 9$. If g is odd, then* $\mathrm{Pic}(\overline{S}_g^+)$ *(resp.* $\mathrm{Pic}(\overline{S}_g^-)$*) is freely generated over $\mathbb{Q}$ by the classes μ^+, α_i^+, β_i^+, $i = 0, \dots, (g-1)/2$ (resp. by the classes μ^-, α_i^-, β_i^-, $i = 0, \dots, (g-1)/2$). If g is even, then* $\mathrm{Pic}(\overline{S}_g^+)$ *(resp.* $\mathrm{Pic}(\overline{S}_g^-)$*) is freely generated over $\mathbb{Q}$ by the classes μ^+, α_i^+, β_i^+, $i = 0, \dots, g/2$ (resp. by the classes μ^-, α_i^-, $i = 0, \dots, g/2$, and β_i^-, $i = 0, \dots, g/2 - 1$).*

Proof. By Lemma 1, we may use the exact sequence

$$A_{3g-4}(\overline{S}_g \setminus S_g) \to A_{3g-4}(\overline{S}_g) \to A_{3g-4}(S_g) \to 0$$

to conclude that $\mathrm{Pic}(\overline{S}_g) \otimes \mathbb{Q}$ is generated by the generators of $A_{3g-4}(S_g)$ together with the set of boundary classes of $\overline{S}_g$. From Theorem 2 it follows that $\mathrm{Pic}(S_g) \otimes \mathbb{Q}$ is generated by the classes μ^+ and μ^-, therefore $\mathrm{Pic}(\overline{S}_g) \otimes \mathbb{Q}$ is generated by the classes μ^+, μ^-, α_i^+, α_i^-, β_i^+, β_i^-. By Proposition 1, all these classes are independent, so Corollary 1 is proved.

□

4. Calculation of Kodaira Dimensions

In this section, we calculate the Kodaira dimension of some spin moduli spaces. We recall that the Kodaira dimension is an important birational invariant in the classification of projective varieties. As a first general result, we prove the following

Proposition 2. *Fix g and n non-negative integers, $n > 2 - 2g$. Any irreducible component of $\overline{S}_{g,n}$ is of general type whenever $\overline{\mathcal{M}}_{g,n}$ is.*

Proof. Since any component of $\overline{S}_{g,n}$ is a non-trivial ramified covering of $\overline{\mathcal{M}}_{g,n}$, then the claim follows from [28], Theorem 6.10. □

In particular, $\overline{S}_g$ is of general type for $g \geq 24$. When $n \geq 1$, Logan determines in [26] integers $\widetilde{n}(g)$, $g \geq 2$, such that $\overline{\mathcal{M}}_{g,n}$ is of general type

when $n \geq \widetilde{n}(g)$. As pointed out in Section 3, when $g = 0$, spin moduli spaces are rational since they are isomorphic to $\overline{\mathcal{M}}_{0,n}$. To tackle the genus one case, we first need to compute the Kodaira dimension of $\overline{\mathcal{M}}_{1,n}$, which does not seem to be thoroughly dealt with in the literature. It turns out that $\kappa(\overline{\mathcal{M}}_{1,n})$ varies with the number of marked points.

As proved in [5], the moduli space of n-pointed genus 1 curves is rational for $n \leq 10$, hence $\kappa(\overline{\mathcal{M}}_{1,n}) = -\infty$. To compute the Kodaira dimension for $n \geq 11$, we first need to express the canonical divisor $K_{\overline{\mathcal{M}}_{1,n}}$ in terms of generators of the rational Picard group of $\overline{\mathcal{M}}_{1,n}$. We briefly recall such generators and some of their relations: for more details the reader is referred, for instance, to [2].

As usual, we denote by λ the first Chern class of the Hodge bundle whose fiber over the element $[C; p_1, \dots, p_n] \in \overline{\mathcal{M}}_{1,n}$ is $H^0(C, \omega_C)$, where ω_C is the dualizing sheaf of C. Next, we denote by ψ_i, $1 \leq i \leq n$, the first Chern class of the line bundle whose fiber over $[C; p_1, \dots, p_n]$ is the cotangent space of C at the (smooth) point p_i. Finally, we denote by δ_{irr} and $\delta_{0,S}$ the classes corresponding to boundary divisors. Here δ_{irr} is the (rational) Poincaré dual of the locus of curves with one non-disconnecting node and n marked points. The class $\delta_{0,S}$, $|S| \geq 2$, corresponds the locus of curves with a disconnecting node whose removal creates two connected components: one of genus 0 with the marked points labelled by the elements of S and the other one of genus 1 with the marked points labelled by the elements of S^c. Last, note that by κ_m and $\widetilde{\kappa}_m$, $m \geq 1$ we mean the Mumford classes as described in [3].

We finally recall that $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$ is isomorphic to the Picard group of the moduli stack of n-pointed genus g stable curves. We shall first compute the canonical class of this stack so to deduce $K_{\overline{\mathcal{M}}_{1,n}}$.

Proposition 3. *For any non-negative integer $n \geq 3$,*

$$K_{\overline{\mathcal{M}}_{1,n}} = (n-11)\lambda + (n-3)\delta_{0,\{1,\dots,n\}} \tag{1}$$

$$+ \sum_{\substack{|S|\geq 2 \\ S\subset\{1,\dots,n\}}, S\neq\{1,\dots,n\}} (|S|-2)\,\delta_{0,S}. \tag{2}$$

Proof. Suppose $\rho : \mathcal{C} \to B$ is a family of n-pointed genus 1 curves over a smooth base B and with general smooth fiber. Let $\sigma_1, \dots, \sigma_n$ denote the n canonical sections and define D to be the sum of the divisors corresponding to them. As in [21], we apply Grothendieck-Riemann-Roch Theorem to the bundle $\Omega^1_{\mathcal{C}/B}(D) \otimes \omega_{\mathcal{C}/B}$, where $\Omega^1_{\mathcal{C}/B}$ is the sheaf of relative Kähler differentials and $\omega_{\mathcal{C}/B}$ is the relative dualizing sheaf of ρ. Therefore,

$$\begin{aligned} K &= c_1(\rho_*(\Omega^1_{\mathcal{C}/B}(D) \otimes \omega_{\mathcal{C}/B})) \\ &= \rho_*\left(Td_2^\vee(\Omega^1_{\mathcal{C}/B}) + ch_2(\Omega^1_{\mathcal{C}/B}(D) \otimes \omega_{\mathcal{C}/B})\right) \\ &+ \rho_*\left(Td_1^\vee(\Omega^1_{\mathcal{C}/B}) ch_1(\Omega^1_{\mathcal{C}/B}(D) \otimes \omega_{\mathcal{C}/B})\right), \end{aligned} \tag{3}$$

where $Td_j^\vee$ and ch_j denote the degree j term of the Todd class $Td^\vee$ and of the Chern character ch, respectively.

If η is the class of the locus of nodes of fibers of $\mathcal{C}$ over B, we have

$$Td_1^{\vee}(\Omega^1_{\mathcal{C}/B}) = -\frac{1}{2}c_1(\omega_{\mathcal{C}/B}),$$

$$Td_2^{\vee}(\Omega^1_{\mathcal{C}/B}) = \frac{1}{12}\left(c_1^2(\omega_{\mathcal{C}/B}) + \eta\right).$$

Analogously,

$$ch_1(\Omega^1_{\mathcal{C}/B}(D) \otimes \omega_{\mathcal{C}/B}) = c_1(\omega_{\mathcal{C}/B}(D)) + c_1(\omega_{\mathcal{C}/B}),$$

$$\begin{aligned} ch_2(\Omega^1_{\mathcal{C}/B}(D) \otimes \omega_{\mathcal{C}/B})) &= \frac{1}{2}c_1^2(\omega_{\mathcal{C}/B}) - \eta \\ &+ c_1(\omega_{\mathcal{C}/B}(D))c_1(\omega_{\mathcal{C}/B}) + \frac{1}{2}c_1^2(\omega_{\mathcal{C}/B}(D)). \end{aligned}$$

By the definition of κ_1 and $\widetilde{\kappa}_1$, we have

$$\begin{aligned} K &= \frac{1}{12}\widetilde{\kappa}_1 + \frac{1}{2}\rho_*\left(c_1(\omega_{\mathcal{C}/B}(D))c_1(\omega_{\mathcal{C}/B})\right) \\ &- \frac{11}{12}\rho_*(\eta) + \frac{1}{2}\kappa_1. \end{aligned} \tag{4}$$

On the other hand, by [4],

$$\kappa_1 = \widetilde{\kappa}_1 + \sum_{i=1}^{n} \psi_i,$$

and

$$\frac{1}{2}\rho_*\left(c_1(\omega_{\mathcal{C}/B}(D))c_1(\omega_{\mathcal{C}/B})\right) = \frac{1}{2}\widetilde{\kappa}_1 + \sum_{i=1}^{n} \psi_i.$$

Therefore, by Example 2.1 in [6], we have

$$K = 13\lambda + \sum_{i=1}^{n} \psi_i - 2\delta, \tag{5}$$

where

$$\delta := \delta_{irr} + \sum_{\substack{|S|\geq 2 \\ S\subset\{1,\ldots,n\}}} \delta_{0,S}.$$

Moreover, in genus 1 (see [3]), this can be rewritten as

$$K = (n-11)\lambda + \sum_{\substack{|S|\geq 2 \\ S\subset\{1,\ldots,n\}}} (|S|-2)\,\delta_{0,S}.$$

Since the map from the moduli stack of n-pointed, $n \geq 3$, genus 1 curves to the (coarse) moduli space is ramified along the divisor $\delta_{0,\{1,\ldots,n\}}$, the claim follows.

□

***Remark* 1.** When $g = 1$ and $n = 1$, the (coarse) moduli space is isomorphic to $\mathbb{P}^1$, so the canonical class is known. When $g = 1$, and $n = 2$, then (5) simplifies to -9λ. Analogously to the case $n \geq 3$, the canonical class is $-9\lambda - \delta_{0,\{1,2\}}$.

***Remark* 2.** In [26], a formula for the canonical divisor of $\overline{\mathcal{M}}_{g,n}$ is given. The proof relies on the corresponding formula for $K_{\overline{\mathcal{M}}_g}$, the canonical divisor of $\overline{\mathcal{M}}_g$, obtained in [22] only for $g \geq 2$ via Grothendieck-Riemann-Roch Theorem. Up to the authors' knowledge, an analogous formula in genus 1 is not explicitely stated in the literature.

We can now complete the computation of $\kappa(\overline{\mathcal{M}}_{1,n})$ for each $n \geq 1$. In fact, the following holds.

Theorem 3. *We have*

$$\kappa(\overline{\mathcal{M}}_{1,n}) = \begin{cases} 0 & n = 11, \\ 1 & n \geq 12. \end{cases}$$

Proof. By Proposition 3, $K_{\overline{\mathcal{M}}_{1,11}}$ is an effective divisor, hence $\kappa(\overline{\mathcal{M}}_{1,11}) \geq 0$. On the other hand, $\mathcal{M}_{1,11}$ is birational to a hypersurface X of $(\mathbb{P}^2)^6$ of multidegree $(3, \ldots, 3)$ (see Remark 1.2.4 in [5]). By adjunction, we obtain that K_X is trivial; in order to compute the Kodaira dimension of X, let $f : Y \to X$ be the normalization map. We have $K_Y = f^* K_X - \Delta$, where the conductor Δ is an effective divisor. It follows that $\kappa(\overline{\mathcal{M}}_{1,11}) = \kappa(X) = \kappa(Y) \leq 0$, so the case $n = 11$ is over.

Next, again by Proposition 3, $K_{\overline{\mathcal{M}}_{1,n}}$, $n \geq 12$, is the sum of two effective divisors, i.e., $L := (n-11)\lambda$ and

$$E := (n-3)\delta_{0,\{1,\ldots,n\}} + \sum_{\substack{|S| \geq 2 \\ S \subset \{1,\ldots,n\}}, S \neq \{1,\ldots,n\}} (|S| - 2)\, \delta_{0,S}.$$

Therefore, the Kodaira dimension of $\overline{\mathcal{M}}_{1,n}$ is greater than or equal to the Iitaka dimension of the divisor L. On the other hand, let

$$\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1} \tag{6}$$

be the morphism which forgets the last $n-1$ points and passes to the stable model. If λ_1 denotes the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{1,1}$, then we have $\lambda = \pi^*(\lambda_1)$ (see for instance [2], (6)). Moreover, since λ_1 is ample on $\overline{\mathcal{M}}_{1,1}$, we obtain

$$\kappa(\overline{\mathcal{M}}_{1,n}, L) \geq \kappa(\overline{\mathcal{M}}_{1,1}, (n-11)\lambda_1) = 1.$$

This proves that $\kappa(\overline{\mathcal{M}}_{1,n}) \geq 1$. However, the fiber of $[C;p] \in \overline{\mathcal{M}}_{1,n}$ under π can be viewed as the quotient by a finite group of an open Zariski subset of the product

$$\underbrace{C \times \ldots \times C}_{\text{(n-1) times}},$$

hence $\kappa(\pi^{-1}([C;p])) \leq 0$. By Theorem 6.12 in [28], it follows that

$$\kappa(\overline{\mathcal{M}}_{1,n}) \leq \kappa(\pi^{-1}([C;p])) + \dim(\overline{\mathcal{M}}_{1,1}) \leq 1.$$

Thus the claim is completely proved.

□

Corollary 2. *For any $n \geq 1$, $\overline{\mathcal{M}}_{1,n}$ is never of general type.*

Next, we turn to moduli spaces of pointed spin curves of genus 1. Recall from Section 3 that $\overline{S}_{1,n}$ is the compactification *à la* Deligne-Mumford of the moduli space of n-pointed smooth elliptic curves with a theta-characteristic. Notice that $\overline{S}_{1,n}$ is the disjoint union of $\overline{S}^+_{1,n}$ and $\overline{S}^-_{1,n}$, which correspond to even and odd theta-characteristics, respectively. However, since over an elliptic curve there is only one odd theta-characteristic (namely, the structural sheaf), there is a natural isomorphism $\overline{S}^-_{1,n} \cong \overline{\mathcal{M}}_{1,n}$. ¿From now onwards, we thus focus our attention on $\overline{S}^+_{1,n}$.

In order to prove that $\kappa(\overline{S}^+_{1,n}) = -\infty$ for $n \leq 10$, we are going to show that $\overline{S}^+_{1,n}$ is uniruled whenever $\overline{\mathcal{M}}_{1,n}$ is. Indeed, the following holds:

Lemma 2. *Let $p : \overline{S}^+_{1,n} \to \overline{\mathcal{M}}_{1,n}$ denote the natural projection. If C is a rational curve in $\overline{\mathcal{M}}_{1,n}$, then there exists a rational curve D in $\overline{S}^+_{1,n}$ such that $p(D) = C$.*

Proof. The proof is by induction on n.

If $n = 1$, then we have $\overline{\mathcal{M}}_{1,n} \cong \mathbb{P}^1$ and $\overline{S}^+_{1,n} \cong \mathbb{P}^1$, so in this case the stated property is obvious.

Assume, now, $n > 1$. It is easy to check that

$$\overline{S}^+_{1,n} = \overline{\mathcal{M}}_{1,n} \times_{\overline{\mathcal{M}}_{1,n-1}} \overline{S}^+_{1,n-1}.$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[dr] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \overline{S}^+_{1,n} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,n} \arrow[d, "\pi"] \\
& \overline{S}^+_{1,n-1} \arrow[r] & \overline{\mathcal{M}}_{1,n-1}
\end{tikzcd}$$

Diagram 1: Uniruledness of spin moduli spaces for $g = 1, n \leq 10$.

In Diagram 1, f exists by hypothesis and g exists by the inductive assumption (just notice that $\pi \circ f(\mathbb{P}^1)$ is not a point since the fibers of π does not contain rational curves). Hence the claim follows from the universal property of the fibered product.

□

Finally, we consider the case $n \geq 12$.

Proposition 4. *Let $n \geq 12$ be a non-negative integer. Then the Kodaira dimension of $\overline{S}^+_{1,n}$ is 1.*

Proof. Since the natural projection

$$p : \overline{S}^+_{1,n} \longrightarrow \overline{\mathcal{M}}_{1,n}$$

is a surjective map between normal varieties of the same dimension, it follows that

$$\kappa(\overline{S}^+_{1,n}) \geq \kappa(\overline{\mathcal{M}}_{1,n}) = 1$$

(see [28], Theorem 6.10).

On the other hand, the fiber of the forgetful map

$$\overline{S}^+_{1,n} \longrightarrow \overline{S}^+_{1,1}$$

is precisely the same as that of the morphism (6). Hence, exactly as in the proof of Theorem 3, we can deduce that

$$\kappa(\overline{S}^+_{1,n}) \leq 0 + \dim(\overline{S}^+_{1,1}) \leq 1.$$

Thus the proof follows.

□

By the same arguments as in Proposition 4, we get the estimate

$$0 \leq \kappa(\overline{S}^+_{1,11}) \leq 1.$$

Another reason why $\kappa(\overline{S}^+_{1,11}) \geq 0$ is given by the following result.

Proposition 5. *There exists a nonzero holomorphic form of degree* 11 *on* $\overline{S}^+_{1,n}$.

Proof. Since the natural projection

$$p : \overline{S}^+_{1,n} \longrightarrow \overline{\mathcal{M}}_{1,n}$$

is a surjective map between normal varieties of the same dimension, the induced map

$$p^* : H^{11,0}(\overline{\mathcal{M}}_{1,n}, \mathbb{Q}) \rightarrow H^{11,0}(\overline{S}^+_{1,n}, \mathbb{Q})$$

is injective.

□

We end this section with a couple of natural questions.

***Question* 1.** Is the Kodaira dimension of $\overline{S}^+_{1,11}$ zero ?

***Question* 2.** Is any irreducible component of $\overline{S}_{g,n}$ unirational whenever the corresponding moduli space $\overline{\mathcal{M}}_{g,n}$ is ?

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